

Quantum entanglement, Calabi-Yau manifolds, and noncommutative algebraic geometry

Shinnosuke Okawa and Kazushi Ueda

Abstract

We relate SLOCC equivalence classes of qudit states to moduli spaces of Calabi-Yau manifolds equipped with a collection of line bundles. The cases of 3 qutrits and 4 qubits are also related to noncommutative algebraic geometry.

1 Introduction

A *qudit* is a d -level quantum mechanical system. The $d = 2$ case, called a *qubit*, is the analogue of a classical bit, and plays a fundamental role in quantum information theory. An essential feature in quantum computing is the existence of *entanglement*. A state $|\psi\rangle \in V_1 \otimes \cdots \otimes V_n$ in an n -qudit system is *separable* if there are states $|\psi^{(1)}\rangle \in V_1, \dots, |\psi^{(n)}\rangle \in V_n$ such that $|\psi\rangle = |\psi^{(1)}\rangle \otimes \cdots \otimes |\psi^{(n)}\rangle$. A state which is not separable is *entangled*.

According to [DVC00], two states $|\psi\rangle$ and $|\psi'\rangle$ are related by a sequence of *stochastic local operations and classical communication (SLOCC)* if there are invertible operators $A^{(1)} \in GL(V_1), \dots, A^{(n)} \in GL(V_n)$ such that $|\psi'\rangle = (A^{(1)} \otimes \cdots \otimes A^{(n)}) |\psi\rangle$. Hence the classification of quantum mechanical states up to SLOCC is equivalent to the classification of $GL(V_1) \times \cdots \times GL(V_n)$ -orbits in $V_1 \otimes \cdots \otimes V_n$. Separable states constitute a single orbit, and all other orbits are entangled.

In this paper, we relate the classification of quantum mechanical states up to SLOCC to the moduli space of algebraic varieties with additional structures. We will deal only with the case $\dim V_1 = \cdots = \dim V_n = d$ for the sake of simplicity, although the same idea works in greater generality. Theorem 2.1 shows that the moduli stack of n -qudit states is birational to the moduli stack of pairs $(Y, (L_1, \dots, L_{n-1}))$ of a Calabi-Yau manifold Y of dimension $(n-1)(d-1)-d$ and a collection (L_1, \dots, L_{n-1}) of line bundles on Y satisfying some conditions.

Note that Y is an elliptic curve if and only if $(n, d) = (3, 3)$ or $(4, 2)$. Elliptic curves have already appeared in several papers in quantum information theory, such as [LT03, BLTV04, Lév11] to name a few. In the case of $(n, d) = (3, 3)$, Theorems 2.1 and 3.1 show that general SLOCC equivalence classes of 3-qutrit states are in one-to-one correspondence with isomorphism classes of triples (C, L_1, L_2) , where C is an elliptic curve and L_1, L_2 are line bundles of degree three on C . This case is also related to non-commutative projective planes, which is a geometric incarnation of three-dimensional Sklyanin algebras [Sk182, ATVdB90]. In the case of $(n, d) = (4, 2)$, Theorems 2.1 and 3.3 show that general SLOCC equivalence classes of 4-qubit states are in one-to-one correspondence with isomorphism

classes of quadruples (C, L_1, L_2, L_3) , where C is an elliptic curve and L_1, L_2, L_3 are line bundles of degree two on C . This case is related to non-commutative quadric surfaces [VdB11]. These moduli spaces are also studied in [BH] from a different point of view.

As a higher-dimensional example, consider the case $(n, d) = (5, 2)$. Theorem 2.1 together with [BHK, Theorem 7.1] shows that general SLOCC equivalence classes of 5-qubit states are in one-to-one correspondence with isomorphism classes of M -polarized K3 surfaces, where $M = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_4$ is a lattice of rank four whose intersection matrix is given by $(e_i, e_j) = 2 - 2\delta_{ij}$.

The idea of using algebraic geometry to classify SLOCC equivalence classes of entangled states also appears in [Lév06, HLT12]. Classification of 4-qubit entanglement is initiated in [VDDMV02], and related to string theory in [BDD⁺10]. See e.g. [BDL12] and references therein for subsequent development.

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2 Moduli stacks

For a pair (d, n) of integers greater than one, we define the category $\mathcal{M}_{d,n}$ as follows:

- An object $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}))$ consists of
 - a smooth morphism $\varphi: \mathcal{Y} \rightarrow S$ of schemes, and
 - a collection $(\mathcal{L}_1, \dots, \mathcal{L}_{n-1})$ of line bundles on \mathcal{Y}

such that

- the relative canonical sheaf $\omega_{\mathcal{Y}/S}$ is trivial,
- $\varphi_*(\mathcal{L}_i)$ for $i = 1, \dots, n-1$ are locally-free sheaves of rank d ,
- the composition morphism

$$\mu: \varphi_*(\mathcal{L}_1) \otimes \cdots \otimes \varphi_*(\mathcal{L}_{n-1}) \rightarrow \varphi_*(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{n-1}) \quad (2.1)$$

is surjective with the kernel of rank d , and

- the natural morphism

$$\mathcal{Y} \rightarrow \mathbb{P}(\varphi_*\mathcal{L}_1) \times_S \cdots \times_S \mathbb{P}(\varphi_*\mathcal{L}_{n-1}) \quad (2.2)$$

is a closed embedding, whose image is a complete intersection over S .

- A morphism from $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}))$ to $(\varphi': \mathcal{Y}' \rightarrow S', (\mathcal{L}'_1, \dots, \mathcal{L}'_{n-1}))$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\phi} & \mathcal{Y}' \\ \varphi \downarrow & & \downarrow \varphi' \\ S & \xrightarrow{\bar{\phi}} & S' \end{array} \quad (2.3)$$

such that

- $\bar{\phi}: S \rightarrow S'$ is a morphism of schemes,
- $\phi: \mathcal{Y} \rightarrow \mathcal{Y}'$ induces an isomorphism $\mathcal{Y} \xrightarrow{\sim} S \times_{S'} \mathcal{Y}'$ of schemes, and
- $\phi^* \mathcal{L}'_i \cong \mathcal{L}_i$ for any $i = 1, \dots, n-1$.

The forgetful functor $\mathcal{M}_{d,n} \rightarrow \mathcal{S}ch/\mathbb{C}$ sending $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}))$ to S makes $\mathcal{M}_{d,n}$ into a category fibered in groupoids.

Let V_1, \dots, V_n be vector spaces of dimension d . The group $G = GL(V_1) \times \dots \times GL(V_n)$ acts naturally on $\mathcal{R} = V_1 \otimes \dots \otimes V_n$, and the quotient stack will be denoted by

$$\mathcal{Q}_{d,n} = [\mathcal{R}/G]. \quad (2.4)$$

This is an Artin stack of finite type over \mathbb{C} . As a category, it is defined as follows:

- An object (S, \mathcal{P}, ψ) consists of
 - a scheme S ,
 - a principal G -bundles $\pi: \mathcal{P} \rightarrow S$, and
 - a G -equivariant morphism $\psi: \mathcal{P} \rightarrow \mathcal{R}$.
- A morphism $(\phi, \bar{\phi})$ consists of
 - a morphism $\bar{\phi}: S \rightarrow S'$ of schemes, and
 - an isomorphism $\phi: \mathcal{P} \xrightarrow{\sim} S \times_{S'} \mathcal{P}'$ of principal G -bundles

such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\psi} & \mathcal{R} \\ \phi \downarrow & & \uparrow \pi' \\ S \times_{S'} \mathcal{P}' & \xrightarrow{\text{pr}_2} & \mathcal{P}' \end{array} \quad (2.5)$$

is commutative.

Theorem 2.1. *The category $\mathcal{M}_{d,n}$ is an Artin stack which is birational to \mathcal{Q} .*

Proof. Let $R = \mathbf{S}(\mathcal{R})$ be the symmetric algebra over the vector space \mathcal{R} . For an element $\eta \in \mathcal{R}$, let $V_\eta \subset V_1 \otimes \dots \otimes V_{n-1}$ be the image of η considered as an element of $V_1 \otimes \dots \otimes V_n \cong \text{Hom}(V_n^\vee, V_1 \otimes \dots \otimes V_{n-1})$. Let further $I_\eta \subset \mathbf{S}(V_1) \otimes \dots \otimes \mathbf{S}(V_{n-1})$ be the ideal generated by V_η , and Y_η be the subscheme of $\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_{n-1})$ defined by I_η . The open subscheme of \mathcal{R} consisting of elements η such that

- V_η is d -dimensional, and
- Y_η is smooth,

will be denoted by \mathcal{R}^{sm} . The G -action on \mathcal{R} preserves \mathcal{R}^{sm} , and we write the corresponding quotient stack as $\mathcal{Q}^{\text{sm}} = [\mathcal{R}^{\text{sm}}/G]$. We will show an equivalence

$$\mathcal{M}_{d,n} \cong \mathcal{Q}^{\text{sm}} \quad (2.6)$$

of categories fibered in groupoids.

We first define a functor $\Psi: \mathcal{M}_{d,n} \rightarrow \mathcal{Q}^{\text{sm}}$ as follows: For an object $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}))$ of the category $\mathcal{M}_{d,n}$, choose an open cover $S = \bigcup_{\lambda \in \Lambda} S_\lambda$ of S such that $\varphi_*(\mathcal{L}_i)|_{S_\lambda}$ is trivial for any $i = 1, \dots, n-1$ and $\lambda \in \Lambda$. Choose a trivialization $\phi_{i,\lambda}: \varphi_*(\mathcal{L}_i)|_{S_\lambda} \xrightarrow{\sim} V_i \otimes \mathcal{O}_{S_\lambda}$ and use this to identify $\text{Ker}(\mu)|_{S_\lambda}$ with the subsheaf of $V_1 \otimes \dots \otimes V_{n-1} \otimes \mathcal{O}_{S_\lambda}$. We may assume that S_λ is sufficiently small so that $\text{Ker}(\mu)|_{S_\lambda}$ is a free sheaf. A choice of a trivialization $\phi_n: \text{Ker}(\mu)|_{S_\lambda} \xrightarrow{\sim} V_n^\vee \otimes \mathcal{O}_{S_\lambda}$ gives a morphism $\bar{\psi}_\lambda: S_\lambda \rightarrow \mathcal{R}$, which corresponds the morphism of locally free sheaves

$$(\otimes_{i=1}^{n-1} \varphi_{i,\lambda}) \circ \varphi_{n,\lambda}^{-1}: V_n^\vee \otimes \mathcal{O}_{S_\lambda} \hookrightarrow V_1 \otimes \dots \otimes V_{n-1} \otimes \mathcal{O}_{S_\lambda}.$$

For another open subscheme S_ρ , the restrictions $\bar{\psi}_\lambda|_{S_\lambda \cap S_\rho}: S_\lambda \cap S_\rho \rightarrow \mathcal{R}$ and $\bar{\psi}_\rho|_{S_\lambda \cap S_\rho}: S_\lambda \cap S_\rho \rightarrow \mathcal{R}$ are related by a change of trivializations ϕ_i for $i = 1, \dots, n$. Since a change of trivialization is given by the action of the group $G = GL(V_1) \times \dots \times GL(V_n)$, one can form a principal G -bundle \mathcal{P} on S by gluing $S_\lambda \times G$ by twisting by the action of G , in such a way that $\bar{\psi}_\lambda$ lifts to a G -equivariant morphism $\psi: \mathcal{P} \rightarrow \mathcal{R}$. The smoothness of the morphism $\varphi: \mathcal{Y} \rightarrow S$ implies that the image of ψ lies in \mathcal{R}^{sm} . The action of Ψ on morphisms is defined in the obvious way.

The functor $\Phi: \mathcal{Q}^{\text{sm}} \rightarrow \mathcal{M}_{d,n}$ in the other direction is defined as follows: Let $\mathcal{Y}_{\mathcal{R}^{\text{sm}}}$ be the subscheme of $\mathcal{R}^{\text{sm}} \times \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_{n-1})$ whose fiber over $\eta \in \mathcal{R}^{\text{sm}}$ is the subscheme $Y_\eta \subset \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_{n-1})$. For an object (S, \mathcal{P}, ψ) of \mathcal{Q}^{sm} , the subscheme $\mathcal{Y}_{\mathcal{P}} = \mathcal{Y}_{\mathcal{R}^{\text{sm}}} \times_{\mathcal{R}^{\text{sm}}} \mathcal{P}$ of $\mathcal{P} \times \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_{n-1})$ has an action of G , which makes the projection $\mathcal{Y}_{\mathcal{P}} \rightarrow \mathcal{P}$ into a G -equivariant morphism. Since G -equivariance is the cocycle condition with respect to the coequalizer diagram

$$\mathcal{P} \times G \rightrightarrows \mathcal{P} \rightarrow S, \quad (2.7)$$

the subscheme $\mathcal{Y}_{\mathcal{P}} \subset \mathcal{P} \times \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_{n-1})$ descends to a subscheme $\mathcal{Y} \subset S \times \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_{n-1})$. Let $\varphi: \mathcal{Y} \rightarrow S$ be the projection to S . The line bundles \mathcal{L}_i for $i = 1, \dots, n-1$ is defined as the pull-back of the tautological line bundle $\mathcal{O}_{\mathbb{P}(V_i)}(1)$ on $\mathbb{P}(V_i)$. Then $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}))$ gives an object of $\mathcal{M}_{d,n}$. The action of Φ on morphisms is defined in the obvious way.

The image of an object $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}))$ of $\mathcal{M}_{d,n}$ by the functor $\Phi \circ \Psi$ is obtained as the zero of

$$\mu: \varphi_*(\mathcal{L}_1) \otimes \dots \otimes \varphi_*(\mathcal{L}_{n-1}) \rightarrow \varphi_*(\mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_{n-1}) \quad (2.8)$$

in $\mathbb{P}(\varphi_*(\mathcal{L}_1)) \otimes_S \dots \otimes_S \mathbb{P}(\varphi_*(\mathcal{L}_{n-1}))$. This is a complete intersection over S of degree $(\mathbf{1}^d) = (\mathbf{1}, \dots, \mathbf{1})$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^{n-1}$. This contains the scheme \mathcal{Y} , which must coincide with \mathcal{Y} since \mathcal{Y} is a complete intersection; if \mathcal{Y} is strictly smaller, then \mathcal{Y} cannot have the trivial canonical bundle by the adjunction formula. This shows that $\Phi \circ \Psi$ is isomorphic to the identity functor. The isomorphism of $\Psi \circ \Phi$ with the identity functor follows immediately from the construction. Hence Ψ and Φ are quasi-inverse to each other, and Theorem 2.1 is proved. \square

Let $H = \{(\lambda_1 \text{id}_{V_1}, \dots, \lambda_n \text{id}_{V_n}) \in G \mid \lambda_1 \cdots \lambda_n = 1\}$ be the generic stabilizer of the G -action on \mathcal{R} , and $G' = G/H$ be the quotient group acting effectively on \mathcal{R} . Let further $S = R^{G'}$ be the invariant ring of R with respect to the action of G' . The ring S inherits the natural grading coming from the standard grading of R as a symmetric algebra. The *geometric invariant theory (GIT) quotient* of \mathcal{R} by G is defined as $Q^{\text{ss}} = \text{Proj } S$ [MFK94]. A point $x \in \mathcal{R}$ is *semi-stable* if there is a G' -invariant polynomial $s \in R^{G'}$ such that $s(x) \neq 0$. Two G -orbits O and O' of \mathcal{R}^{ss} are related by the *closure equivalence* if the closures of O and O' intersect. Geometric points of the GIT quotient are in one-to-one correspondence with closure equivalence classes of G -orbits in \mathcal{R}^{ss} . The GIT quotient Q^{ss} is the best approximation of the quotient stack $\mathcal{Q}^{\text{ss}} = [\mathcal{R}^{\text{ss}}/G]$ by a scheme. The importance of GIT in quantum entanglement is first pointed out by Klyachko [Kly].

Proposition 2.2. *The moduli stack $\mathcal{M}_{d,n}$ is birational to \mathcal{Q}^{ss} .*

Proof. Since the hyperdeterminant of format $2 \times \cdots \times 2$ is non-trivial and G' -invariant by [GKZ94, Theorems 14.1.3 and 14.1.4], the semi-stable locus \mathcal{R}^{ss} is non-empty. Since semi-stability is an open condition, \mathcal{R}^{ss} is a non-empty open subscheme of \mathcal{R} . The theorem of Bertini shows that \mathcal{R}^{sm} is a non-empty open subset of \mathcal{R} . Hence \mathcal{R}^{ss} and \mathcal{R}^{sm} has a common open dense subset, so that the stacks $\mathcal{M}_{d,n} \cong [\mathcal{R}^{\text{sm}}/G]$ and $\mathcal{Q}^{\text{ss}} = [\mathcal{R}^{\text{ss}}/G]$ are birational. \square

Let $[\mathcal{R}^{\text{ss}}/G']$ be the stack obtained from \mathcal{Q}^{ss} by removing the generic stabilizer. The following proposition shows that the forgetful morphism from $\mathcal{M}_{d,n}$ to the moduli space of Calabi-Yau manifolds (without any additional structure) is dominant on an irreducible component:

Lemma 2.3. *One has $\dim H^1(T_Y) = \dim[\mathcal{R}^{\text{ss}}/G']$ if $\dim Y \geq 3$.*

Proof. Let $\pi_i : \mathbb{P} = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_{n-1}) \rightarrow \mathbb{P}(V_i)$ be the i -th projection and set

$$\begin{aligned} \mathcal{O}_{\mathbb{P}}(\mathbf{a}) &= \mathcal{O}_{\mathbb{P}(V_1)}(a_1) \boxtimes \cdots \boxtimes \mathcal{P}_{\mathbb{P}(V_{n-1})}(a_{n-1}) \\ &= \bigotimes_{i=1}^{n-1} \pi_i^* \mathcal{O}_{\mathbb{P}(V_i)}(a_i) \end{aligned}$$

for $\mathbf{a} = (a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$. Let $\{\mathbf{e}_i\}_{i=1}^{n-1}$ be the standard basis of \mathbb{Z}^{n-1} and set $\mathbf{1} = e_1 + \cdots + e_{n-1}$. In the exact sequence

$$0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_{\mathbb{P}|Y} \rightarrow \mathcal{N}_{Y/\mathbb{P}} \rightarrow 0, \quad (2.9)$$

one has $\mathcal{N}_{Y/\mathbb{P}} \cong \mathcal{O}_Y(\mathbf{1})^{\oplus d}$ since Y is a complete intersection of degree $(\mathbf{1}^d)$. By tensoring the Koszul resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d \cdot \mathbf{1}) \rightarrow \cdots \rightarrow \mathcal{O}_{\mathbb{P}}(-2 \cdot \mathbf{1})^{\oplus \binom{d}{2}} \rightarrow \mathcal{O}_{\mathbb{P}}(-\mathbf{1})^{\oplus d} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_Y \rightarrow 0 \quad (2.10)$$

with $\mathcal{O}_Y(\mathbf{1})$ and taking the global section, one obtains

$$\dim H^0(\mathcal{O}_Y(\mathbf{1})) = d^{n-1} - d, \quad (2.11)$$

so that

$$\dim H^0(\mathcal{N}_{Y/\mathbb{P}}) = \dim H^0(\mathcal{O}_Y(\mathbf{1}))^{\oplus d} = d(d^{n-1} - d). \quad (2.12)$$

The pull-back of the Euler sequence on $\mathbb{P}(V_i)$ gives

$$0 \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathbb{P}}(\mathbf{e}_i)^{\oplus d} \rightarrow \pi_i^* \mathcal{T}_{\mathbb{P}(V_i)} \rightarrow 0, \quad (2.13)$$

which together with $\mathcal{T}_{\mathbb{P}} = \bigoplus_{i=1}^{n-1} \pi_i^* \mathcal{T}_{\mathbb{P}(V_i)}$ gives

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}^{\oplus(n-1)} \rightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}}(\mathbf{e}_i)^{\oplus d} \rightarrow \mathcal{T}_{\mathbb{P}} \rightarrow 0. \quad (2.14)$$

By restricting to Y , one obtains

$$0 \rightarrow \mathcal{O}_Y^{\oplus(n-1)} \rightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(\mathbf{e}_i)^{\oplus d} \rightarrow \mathcal{T}_{\mathbb{P}|Y} \rightarrow 0. \quad (2.15)$$

One can show from (2.10) that $H^1(\mathcal{O}_Y(\mathbf{e}_i)) = 0$, so that

$$\dim H^0(\mathcal{T}_{\mathbb{P}|Y}) = (n-1)d^2 - (n-1). \quad (2.16)$$

and

$$H^1(\mathcal{T}_{\mathbb{P}|Y}) \cong H^2(\mathcal{O}_Y^{\oplus(n-1)}). \quad (2.17)$$

One has $H^0(\mathcal{T}_Y) = 0$ if $\dim Y \geq 2$, and $H^2(\mathcal{O}_Y) = 0$ if $\dim Y \geq 3$. The long exact sequence associated with (2.9) gives

$$0 \rightarrow H^0(\mathcal{T}_{\mathbb{P}|Y}) \rightarrow H^0(\mathcal{N}_{Y/\mathbb{P}}) \rightarrow H^1(\mathcal{T}_Y) \rightarrow 0. \quad (2.18)$$

It follows that

$$\begin{aligned} \dim H^1(\mathcal{T}_Y) &= \dim H^0(\mathcal{N}_{Y/\mathbb{P}}) - \dim H^0(\mathcal{T}_{\mathbb{P}|Y}) \\ &= d(d^{n-1} - d) - ((n-1)d^2 - (n-1)) \\ &= d^n - nd^2 + n - 1, \end{aligned}$$

which coincides with

$$\begin{aligned} \dim[\mathcal{R}^{\text{ss}}/G'] &= \dim \mathcal{R} - \dim G' \\ &= d^n - (nd^2 - n + 1) \\ &= d^n - nd^2 + n - 1. \end{aligned}$$

□

3 Noncommutative algebraic geometry

A \mathbb{Z} -algebra is an algebra of the form $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ with the property that the product satisfies $A_{ij}A_{kl} = 0$ for $j \neq k$ and $A_{ij}A_{jk} \subset A_{ik}$. We assume that A_{ii} for any $i \in \mathbb{Z}$ has an element e_i , called the *local unit*, which satisfies $fe_i = f$ and $e_ig = g$ for any $f \in A_{ki}$ and $g \in A_{ij}$. A graded A -module is a right A -module of the form $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $M_iA_{kl} = 0$ for $k \neq i$ and $M_iA_{ij} \subset A_j$. An A -module is a *torsion* if it is a colimit of modules M satisfying $M_i = 0$ for sufficiently large i . Let $P_i = e_iA$ and $S_i = e_iAe_i$ be right A -modules. A \mathbb{Z} -algebra A is *positively graded* if $A_{ij} = 0$ for $i < j$. A positively graded \mathbb{Z} -algebra A is *connected* if $\dim A_{ij} < \infty$ and $A_{ii} = \mathbb{C}e_i$ for any $i, j \in \mathbb{Z}$. A connected \mathbb{Z} -algebra A is *AS-regular* if

- $\dim A_{ij}$ is bounded by a polynomial in $j - i$,
- the projective dimension of S_i is bounded by a constant independent of i ,
- $\sum_{j,k} \dim \operatorname{Ext}_{\operatorname{Gr} A}^j(S_k, P_i) = 1$ for any $i \in \mathbb{Z}$.

An AS-regular \mathbb{Z} -algebra A is a *3-dimensional quadratic AS-regular \mathbb{Z} -algebra* if the minimal resolution of S_i is of the form

$$0 \rightarrow P_{i+3} \rightarrow P_{i+2}^3 \rightarrow P_{i+1}^3 \rightarrow P_i \rightarrow S_i \rightarrow 0. \quad (3.1)$$

An AS-regular \mathbb{Z} -algebra A is a *3-dimensional cubic AS-regular \mathbb{Z} -algebra* if the minimal resolution of S_i is of the form

$$0 \rightarrow P_{i+4} \rightarrow P_{i+3}^2 \rightarrow P_{i+1}^2 \rightarrow P_i \rightarrow S_i \rightarrow 0. \quad (3.2)$$

Let $\operatorname{Qgr} A = \operatorname{Gr} A / \operatorname{Tor} A$ be the quotient abelian category of the abelian category $\operatorname{Gr} A$ of graded A -modules by the Serre subcategory $\operatorname{Tor} A$ consisting of torsion modules. A *noncommutative projective plane* is an abelian category of the form $\operatorname{Qgr} A$ for a 3-dimensional quadratic AS-regular \mathbb{Z} -algebra. A *noncommutative quadric surface* is defined similarly as $\operatorname{Qgr} A$ for a 3-dimensional cubic AS-regular \mathbb{Z} -algebra [VdB11, Definition 3.2].

The classification of 3-dimensional regular quadratic \mathbb{Z} -algebras can be found in [VdB11, Proposition 3.3], which is essentially due to [BP93]. They are divided into the *linear* case and the *elliptic* case. The abelian category $\operatorname{Qgr} A$ is equivalent to $\operatorname{Qcoh} \mathbb{P}^2$ in the linear case. The elliptic cases are classified by triples (C, L_1, L_2) of a curve C and a pair (L_1, L_2) of line bundles such that

- the curve C is embedded as a divisor of degree 3 in \mathbb{P}^2 by global sections of both L_1 and L_2 ,
- $\deg(L_1|_D) = \deg(L_2|_D)$ for every irreducible component D of C , and
- L_1 is not isomorphic to L_2 .

Accordingly, the moduli stack \mathcal{N}_3 of elliptic triples is defined as a category whose object $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \mathcal{L}_2))$ consists of a flat morphism $\varphi: \mathcal{Y} \rightarrow S$ of schemes and a pair $(\mathcal{L}_1, \mathcal{L}_2)$ of line bundles on \mathcal{Y} satisfying the above three conditions on geometric points of S . The morphisms are defined in the same way as that for $\mathcal{M}_{d,n}$.

Theorem 3.1. *The moduli stack $\mathcal{M}_{3,3}$ is birational to the moduli stack \mathcal{N}_3 .*

Proof. Let L be a line bundle on an elliptic curve. Serre duality

$$H^1(L)^\vee \cong H^0(L^\vee) \quad (3.3)$$

shows that $h^1(L) = 0$ if $h^0(L) > 1$, and Riemann-Roch theorem

$$h^0(L) - h^1(L) = \deg L \quad (3.4)$$

shows that $\deg L = 3$ if and only if $h^0(L) = 3$. If this is the case, then L is very ample (cf. e.g. [Har77, Corollary IV.3.2(b)]). It follows that the open substack of $\mathcal{M}_{3,3}$ consisting of objects $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \mathcal{L}_2))$ such that \mathcal{L}_1 is not isomorphic to the tensor product of \mathcal{L}_2 and a pull-back of a line bundle on S is isomorphic to the open substack of \mathcal{N}_3 consisting of objects $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \mathcal{L}_2))$ such that φ is smooth, and Theorem 3.1 is proved. \square

The classification of 3-dimensional cubic AS-regular \mathbb{Z} -algebras is given in [VdB11, Proposition 4.2]. They are divided into the *linear* case and the *elliptic* case. The abelian category $\text{Qgr } A$ is equivalent to $\text{Qcoh } \mathbb{P}^1 \times \mathbb{P}^1$ in the linear case. The elliptic cases are classified by quadruples (C, L_1, L_2, L_3) of a curve C and line bundles L_1, L_2 , and L_3 such that

- the curve C is embedded as a divisor of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by global sections of both (L_1, L_2) and (L_2, L_3) ,
- $\deg(L_1|_D) = \deg(L_3|_D)$ for every irreducible component D of C , and
- L_1 is not isomorphic to L_3 .

The moduli stack \mathcal{N}_4 of elliptic quadruples is defined as a category whose object $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3))$ consists of a flat morphism $\varphi: \mathcal{Y} \rightarrow S$ of schemes and a triple $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ of line bundles on \mathcal{Y} satisfying the above three conditions on geometric point of S .

Lemma 3.2. *Let (L, L') be a pair of line bundles of degree 2 on an elliptic curve C . Then the morphism $\varphi_{(L, L')}: C \rightarrow \mathbb{P}(H^0(L)) \times \mathbb{P}(H^0(L'))$ defined by (L, L') is an embedding if and only if L is not isomorphic to L' .*

Proof. It is clear that $\varphi_{(L, L')}$ factors through the diagonal embedding

$$\mathbb{P}(H^0(L)) \rightarrow \mathbb{P}(H^0(L)) \times \mathbb{P}(H^0(L)), \quad (3.5)$$

and hence can never be an embedding. Consider the composition map

$$\mu: H^0(L) \otimes H^0(L') \rightarrow H^0(L \otimes L'). \quad (3.6)$$

Note that $h^0(L) = h^0(L') = 2$ and $h^0(L \otimes L') = 4$. If the map μ is surjective, then the composition of the morphisms

$$C \xrightarrow{\varphi_{(L, L')}} \mathbb{P}(H^0(L)) \times \mathbb{P}(H^0(L')) \xrightarrow{\text{Segre}} \mathbb{P}(H^0(L) \otimes H^0(L')) \xrightarrow{\mu_*} \mathbb{P}(H^0(L \otimes L')) \quad (3.7)$$

gives the morphism $\varphi_{L \otimes L'}: C \rightarrow \mathbb{P}(H^0(L \otimes L'))$ associated with the line bundle $L \otimes L'$. Here, the second arrow is the Segre embedding, and the third arrow is induced by μ .

Note that the line bundle $L \otimes L'$ has degree 4, and hence very ample, so that $\varphi_{L \otimes L'}$ is an embedding. The fact that this embedding factors through $\varphi_{(L, L')}$ shows that the morphism $\varphi_{(L, L')}$ is an embedding.

Assume that the map μ is not surjective. Choose a basis $\{s, t\}$ of $H^0(L)$. If one takes an element of $\ker \mu$, then it can be written as $s \otimes a + t \otimes b \in \ker \mu$ for some $a, b \in H^0(L')$. It follows that $sa = -tb$ in $H^0(L \otimes L')$, which implies that s divides b . Since $\deg L = \deg L'$, this is possible only if L and L' are isomorphic, and Lemma 3.2 is proved. \square

Theorem 3.3. *The moduli stack $\mathcal{M}_{4,2}$ is birational to the moduli stack \mathcal{N}_4 .*

Proof. Adjunction formula shows that an elliptic curve embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ must be a divisor of bidegree $(2, 2)$. Lemma 3.2 shows that the open substack of $\mathcal{M}_{4,2}$ consisting of objects $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3))$ such that $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 are distinct is isomorphic to the open substack of \mathcal{N}_4 consisting of objects $(\varphi: \mathcal{Y} \rightarrow S, (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3))$ such that φ is smooth. \square

The geometry of the moduli stacks $\mathcal{M}_{3,3}, \mathcal{M}_{4,2}, \mathcal{N}_3, \mathcal{N}_4$ and the corresponding compact moduli schemes will be studied in more detail in [AOU, OU] from the viewpoint of noncommutative algebraic geometry.

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Shinnosuke Okawa

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.

e-mail address : okawa@math.sci.osaka-u.ac.jp

Kazushi Ueda

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.

e-mail address : kazushi@math.sci.osaka-u.ac.jp